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Strategy-Proofness and Arrow's Conditions: Existence and Correspondence Theorems for Voting Procedures and Social Welfare Functions*

MARK ALLEN SATTERTHWAITE

Department of Managerial Economics and Decision Sciences, Graduate School of Management, Northwestern University, Evanston, Illinois 60201

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Consider a committee which must select one alternative from a set of three or more alternatives. Committee members each cast a ballot which the voting procedure counts. The voting procedure is strategy-proof if it always induces every committee member to cast a ballot revealing his preference. I prove three theorems. First, every strategy-proof voting procedure is dictatorial. Second, this paper's strategy-proofness condition for voting procedures corresponds to Arrow's rationality, independence of irrelevant alternatives, non-negative response, and citizens' sovereignty conditions for social welfare functions. Third, Arrow's general possibility theorem is proven in a new manner.

1. INTRODUCTION

Almost every participant in the formal deliberations of a committee realizes that situations may occur where he can manipulate the outcome of the committee's vote by misrepresenting his preferences. For example, a voter in choosing among a Democrat, a Republican, and a minor party candidate may decide to follow the "sophisticated strategy" of voting for his second choice, the Democrat, instead of his "sincere strategy" of voting for his first choice, the minor party candidate, because he thinks that a vote for the minor party candidate would be a wasted vote on a hopeless cause.[1] The fundamental question I ask in this paper is if a committee can eliminate use of sophisticated strategies among its members by constructing a voting procedure that is "strategy-proof" in the sense

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[1] Farquharson [4] introduced the terms sophisticated strategy and sincere strategy.

that under it no committee member will ever have an incentive to use a sophisticated strategy. I prove a negative answer: If a committee is choosing among at least three alternatives, then every strategy-proof voting procedure vests in one committee member absolute power over the committee's choice. In other words, every strategy-proof voting procedure is dictatorial.

This result, which is reminescent of Arrow's general possibility theorem for social welfare functions [1], suggests a second question. What is the relationship between the requirement for voting procedures of strategy-proofness and Arrow's requirements [1] for social welfare functions of rationality, nonnegative response, citizens' sovereignty, and independence of irrelevant alternatives? I show that they are equivalent: a one-to-one correspondence exists between every strategy-proof voting procedure and every social welfare function satisfying Arrow's four requirements. This means that if a social welfare function violates any one of Arrow's requirements, then the voting procedure which is naturally derived from the social welfare function is not strategy-proof. Last, for the third result of the paper, I use the first two results to construct a new proof of Arrow's general possibility theorem.

The questions of this paper are not new. Black [2, p. 182] quotes the vexed retort, "My scheme is only intended for honest men!", which Jean-Charles de Borda, the eighteenth century voting theorist, made when a colleague pointed out how easily his Borda count can be manipulated by sophisticated strategies. More recently Arrow [1, p. 7] suggested that strategy-proofness is an appropriate criterion for evaluating voting procedures. Dummet and Farquharson [3] conjectured in passing that for the case of three or more alternatives no nondictatorial strategy-proof voting procedure exists. By means of distinctly different techniques Gibbard [7] and Satterthwaite [13] independently formalized and proved this conjecture.[2] In addition Zeckhauser [19] proved a similar existence theorem. Vickery [18] and Gibbard [7] speculated about, but did not definitively establish the relationship between strategy-proofness and Arrow's four requirements. Finally, Farquharson [4], Sen [16, pp. 193–194], and Pattanik [9–11] each commented on different aspects of the manipulability of non-dictatorial voting procedures.

This paper has six sections. In Section 2 I formulate the problem and

[2] In my doctoral dissertation [13] I stated Theorem 1 (existence of a strategy proof voting procedure) and proved it using the constructive proof presented in Section 3 of this paper. This work was done independently of Gibbard. Subsequently, an anonymous referee informed me of Gibbard's paper. The statement and proof in Section 4 of Theorem 2 (correspondence of strategy proofness and Arrow's conditions) followed directly from the insight which I gained from reading Gibbard's paper.

establish notation. The next three sections contain in sequence the paper's three results: strategy-proof voting procedures are necessarily dictatorial; a one-to-one correspondence exists between strategy-proof voting procedures and social welfare functions satisfying rationality, nonnegative response, citizens' sovereignty, and independence of irrelevant alternatives; and construction of a new proof of Arrow's general possibility theorem using the first two results. In order to clarify the exposition of these three sections I have made within them the restrictive assumption that indifference is inadmissable. In Section 6 I eliminate this assumption and show how each of the results extend to the general case where indifference between alternatives is admissable.

2. Formulation

Let a *committee* be a set I_n of n, $n \geqslant 1$, individuals whose task is to select a single alternative from an *alternative set* S_m of m elements, $m \geqslant 3$. Each individual $i \in I_n$ has *preferences* R_i which are a weak order on S_m, i.e., R_i is reflexive, complete, and transitive.[3] Thus, if $x, y \in S_m$ and $i \in I_n$, then xR_iy means that individual i either prefers that the committee choose alternative x instead of y or is indifferent concerning which of the two alternatives the committee chooses. Strict preference for x over y on the part of individual i is written as $x\bar{R}_iy$. Thus, $x\bar{R}_iy$ is equivalent to writing xR_iy and $\sim yR_ix$. Indifference is written as xR_iy and yR_ix. Let π_m represent the collection of all possible preferences and let ${\pi_m}^n$ represent the n-fold cartesian product of π_m.

The committee makes its selection of a single alternative by voting. Each individual $i \in I_n$ casts a *ballot* B_i which is a weak order on S_m, i.e., $B_i \in \pi_m$. The ballots are counted by a *voting procedure* v^{nm}. Formally, a voting procedure is a singlevalued mapping whose argument is the ballot set $B = (B_1, ..., B_n) \in {\pi_m}^n$ and whose image is the *committee's choice*, a single alternative $x \in S_m$. Every voting procedure v^{nm} has a domain of ${\pi_m}^n$ and a range of either S_m or some nonempty subset of S_m. Let the range be labeled T_p where p, $1 \leqslant p \leqslant m$, is the number of elements contained in T_p. Given these definitions, let the tetrad $\langle I_n, S_m, v^{nm}, T_p\rangle$ be called the *committee's structure.*

This formulation of the committee decision problem incorporates two assumptions which particularly merit further comment. First, the committee makes only a single decision. This assumption excludes from

[3] The following symbols of mathematical logic are used: $\in$ element of, $\subset$ subset of, $\subsetneq$ strict subset of, $\cup$ union of two sets, $\cap$ intersection of two sets, and $\sim$ not.

consideration such committee behaviors as logrolling which may occur whenever a committee is making a sequence of decisions. Second, the committee selects a single alternative from the alternative set. This contrasts with Arrow's [1] and Sen's [15–17] specification of set valued decision functions. They made that specification because their focus was social welfare where partitioning the alternative set into classes of equal welfare is a useful result. Nevertheless, specification of set valued decision functions (voting rules) is inappropriate here because committees often must choose among mutually exclusive courses of action.[4] For example, a committee can adopt only one budget for a particular activity and fiscal period.

With the basic structure of the committee specified, I can define the concept of a strategy-proof voting procedure. Consider a committee with structure $\langle I_n, S_m, v^{nm}, T_p\rangle$. Individual $i \in I_n$ can manipulate the voting procedure v^{nm} at ballot set $B = (B_1, ..., B_n) \in \pi_m{}^n$ if and only if a ballot $B_i' \in \pi_m$ exists such that

$$v^{nm}(B_1, ..., B_i', ..., B_n)\ \bar{B}_i v^{nm}(B_1, ..., B_i, ..., B_n). \tag{1}$$

Thus, v^{nm} is manipulable at B if an individual $i \in I_n$ can substitute ballot B_i' for B_i and secure a more favorable outcome by the standards of the original ballot B_i. The voting procedure v^{nm} is *strategy-proof* if and only if no $B \in \pi_m{}^n$ exists at which it is manipulable.[5]

This definition has two interpretations. If a voting procedure v^{nm} is not strategy-proof, then a ballot set $B = (B_1, ..., B_i, ..., B_n) \in \pi_m{}^n$ and ballot $B_i' \in \pi_m$ exists such that v^{nm} is manipulable at B. Suppose the ballot B_i faithfully represents the preferences of individual i in the specific sense that $B_i \equiv R_i$. By substituting ballot B_i' for B_i individual i can improve the outcome of the vote according to his own preferences, i.e.,

$$v^{nm}(B_1, ..., B_i', ..., B_n)\ \bar{R}_i v^{nm}(B_1, ..., R_i, ..., B_n). \tag{2}$$

The ballot $B_i \equiv R_i$ is the individual's *sincere strategy* and the ballot $B_i' \neq R_i$ is a *sophisticated strategy.*

[4] Set valued decision functions can give unambiguous choices if they are coupled with a lottery mechanism that randomly selects one alternative from among any sets of tied alternatives. This is the approach which Fishburn [6] and Zeckhauser [19] adopted. I reject this approach here because I think that the use of decision mechanisms with a random element would be politically unacceptable to almost all committees. Gibbard [7] argued in detail in favor of this paper's approach.

[5] I have adapted this definition of strategy-proofness from Schmeidler and Sonnenschein [14]. My earlier definition in [13] is equivalent, but more awkward to use in proofs.

The second interpretation relates to the theory of games. If a voting procedure v^{nm} is strategy-proof, then no situation can arise where an individual $i \in I_n$ can improve the vote's outcome relative to his preferences R_i by employing a sophisticated strategy. Consequently, if v^{nm} is strategy-proof, then every set of sincere strategies $R = (R_1, ..., R_n) \in \pi_m{}^n$ is an equilibrium as defined by Nash [8]. If the voting procedure is not strategy-proof, then there must exist a set of sincere strategies $R = (R_1, ..., R_n) \in \pi_m{}^n$ which is not a Nash equilibrium.

Until this point I have defined the preferences and ballots of committee members to be weak orders over the alternative set. For the purpose of proof this is an inconvenient convention. Therefore, throughout a majority of this paper, I recognize as admissable preferences and ballots only strong orders. Let ρ_m and $\rho_m{}^n$, respectively, label the set of strong orders over S_m and the n-fold cartesian product of ρ_m. Since strong orders exclude the possibility of indifference, if $x, y \in S_m$, $x \neq y$, and $R_i \in \rho_m$, then xR_iy implies $x\bar{R}_iy$ and $\sim yR_ix$. Similarly, if $x, y \in S_m$, $x \neq y$, and $B_i \in \rho_m$, then xB_iy implies $x\bar{B}_iy$ and $\sim yB_ix$. Formally:

RESTRICTION D. *Consider a committee with structure $\langle I_n, S_m, v^{nm}, T_p\rangle$. If this structure is subject to restriction D, then only preference sets $R = (R_1, ..., R_n) \in \rho_m{}^n$ and ballot sets $B = (B_1, ..., B_n) \in \rho_m{}^n$ are admissible.*

A committee subject to Restriction D is called a *strict committee* and its voting procedure is called a *strict voting procedure*. For strict committees the definitions given above must be revised with the substitution of $\rho_m{}^n$ for $\pi_m{}^n$. Thus, a strict voting procedure v^{nm} has a domain of $\rho_m{}^n$ and is strategy-proof if and only if there exists no $B \in \rho_m{}^n$ at which it is manipulable.

My notational conventions for this paper are that the letters B, C, and D represent ballot sets or, if subscripted, individual ballots. The letters U, V, and W represent subsets of S_m or T_p. The letters i and j index the individuals who are committee members and the letters w, x, y, and z represent elements of S_m. Script upper case letters represent collections of voting procedures or social welfare functions. Finally, Ψ and θ represent two functions which appear throughout the remainder of the paper.

The choice function Ψ_W, defined for any $W \subset S_m$, is a mapping from π_m into the nonempty subsets of S_m. It has the property that $x \in \Psi_W(B_i)$ for some $B_i \in \pi_m$ if and only if $x \in W$ and xB_iy for all $y \in W$. In other words, Ψ_W picks out those elements of W which the weak ordering B_i ranks highest. Turning to the function θ_W, let W be a subset of S_m that has

$q \leqslant m$ elements. Define θ_W to be a mapping from π_m to π_q with the property that if $x, y \in W$, $C_i \in \pi_q$, $D_i \in \pi_m$, and $C_i = \theta_W(D_i)$, then $xC_i y$ if and only if $xD_i y$. Thus, θ_W constructs a new weak ordering C_i from D_i by simply deleting those elements of S_m that are not contained in W.

3. Existence Theorem for Voting Procedures

In this section I prove that if a strict voting procedure includes at least three elements in its range and is strategy-proof, then it is dictatorial. A dictatorial voting procedure, as its name implies, vests all power in one individual, the dictator, who determines the committee's choice by his choice of that element of the voting procedure's range which he ranks highest on his ballot. Formally, consider a voting procedure v^{nm} with range T_p. Define for all $B \in \pi_m{}^n$ and for some $i \in I_n$ the function $f_T{}^i(B)$ so that it is singlevalued, has range T_p, and if $f_T{}^i(B) = x$ then $xB_i y$ for all $y \in T_p$. The voting procedure v^{nm} is *dictatorial* if and only if an $i \in I_n$ exists such that $v^{nm}(B) = f_T{}^i(B)$ for all $B \in \pi_m{}^n$. Notice that $f_T{}^i(B)$ is identical to the choice function $\Psi_T(B_i)$ except that $f_T{}^i(B)$ has a tie-breaking property which the set valued $\Psi_T(B_i)$ does not have.

Since I define dictatorial voting procedures with reference to its range T_p, not with reference to the alternative set S_m, two varieties of dictatorial voting procedures are possible. First, *fully dictatorial voting procedures* have as their ranges the full alternative set: $T_p \equiv S_m$. Second, *partially dictatorial voting procedures* have as their ranges proper subsets of the full alternative set: $T_p \Subset S_m$. In other words, if the voting procedure is partially dictatorial, then imposed on the dictator's power is the constraint that he can not pick any $x \in S_m$ such that $x \notin T_p$.

The dictator of a dictatorial voting procedure never has any reason to misrepresent his preferences because the committee's choice is always that element of the range which the dictator ranks first on his ballot. The same is not necessarily true for other individuals. If at the top of his ballot the dictator states that he is indifferent among a group of several alternatives, then the dictatorial voting procedure may resolve the tie by consulting the ballots of the other individuals. If, for example, the Borda count is the method used to count the other individuals' ballots, then the manipulability of the Borda count when the choice is among at least three alternatives may give these individuals an opportunity to manipulate the outcome. Thus, when indifference among alternatives is admissable, dictatoriality is a necessary but not a sufficient condition for strategy-proofness. It is necessary and sufficient when indifference is not admissable.

Theorem 1 is the existence theorem for strict strategy-proof voting procedures. In Section 6 I extend it to the case of non-strict committees.

THEOREM 1. (Gibbard–Satterthwaite). *Consider a strict committee with structure* $\langle I_n, S_m, v^{nm}, T_p\rangle$ *where* $n \geqslant 1$ *and* $m \geqslant p \geqslant 3$. *The voting procedure* v^{nm} *is strategy-proof if and only if it is dictatorial.*

This is formally a possibility theorem, but its substance is that of an impossibility theorem because no committee with democratic ideals will use a dictatorial voting procedure. Such a voting procedure vests all power in one individual, an unacceptable distribution.

The theorem limits itself to the interesting case where the voting procedure's range includes at least three alternatives. If its range contains less than three elements, then a trivial result is that two more types of strategy proof voting procedures exist: imposed procedures and twin alternative voting procedures.[6] These two types are of little interest because committees usually must select among three or more alternatives.

An imposed voting procedure is one where no individual's ballot has any influence on the decision. Thus, a voting procedure is *imposed* if there exists a $x \in S_m$ such that $v^{nm}(B) = x$ for all $B \in \pi_m{}^n$. Imposed voting procedures are strategy-proof because no individual's choice of strategy-affects the committee's choice.[7] Twin alternative voting procedures have ranges that are limited to only two elements of the alternative set. Formally, if a set $T_2 = (x, y) \subset S_m$, $x \neq y$, exists such that $v^{nm}(B) \in T_2$ for all $B \in \pi_m{}^n$, then v^{nm} is a *twin alternative voting procedure*. An example of a strategy-proof twin alternative voting procedure for a committee considering the alternative set $S_4 = (w, x, y, z)$ is defined by the rule: select alternative x or z depending on which is ranked higher on a majority of the committee members' ballots. Alternatives w and y are excluded no matter how the committee votes. This twin alternative voting procedure is strategy proof because each individual has only two choices: vote for or against his preferred alternative. Obviously, in this case, he has every reason to vote for his preferred alternative no matter what his subjective

[6] Another class of strategy proof committee decision rules exist, but they do not satisfy our definition of a voting procedure because they involve a lottery. Let a lottery be held among the committee members' ballots with each ballot having an equal opportunity of winning. The top ranked alternative on the winning ballot is then declared the committee's choice. This rule is strategy-proof, but its probabilistic nature would undoubtedly offend most committees. For a full discussion of lotteries as strategy-proof social choice mechanisms see Zeckhauser [19].

[7] One may argue here that individuals have no incentive to play any strategy at all, whether sophisticated or sincere. Yet an imposed voting procedure is strategy-proof according to the definitions established above.

estimate of how the other individuals will vote is. Nevertheless, a twin alternative voting procedure is not necessarily strategy-proof because conceivably it might perversely count a vote for one included alternative as a vote for the other included alternative.

The proof presented here of Theorem 1 is by construction. I first show that the theorem is true for the case where $n = 1$ and $m = 3$. Next I prove that, where $m = 3$, if the theorem is true for any $n = n'$, then it is true for $n = n' + 1$. This sets up an inductive chain and therefore, in the $m = 3$ case, the theorem is true for all $n \geqslant 1$. Finally, given any arbitrary $n \geqslant 1$, an inductive chain on m can be set up to establish the theorem's validity for $m > 3$. This proof is direct and is not based on Arrow's impossibility theorem. In both these respects it is different from Gibbard's proof [7] of this same theorem.

A necessary preliminary before beginning the proof's substance is to define weak and strong alternative-excluding voting procedures. A strict voting procedure v^{nm} is *weak alternative-excluding* if and only if there exists at least one alternative $x \in S_m$ such that $v^{nm}(B) \neq x$ for all $B \in \rho_m{}^n$. Thus, v^{nm} is weak alternative-excluding if and only if $T_p \Subset S_m$, i.e., its range must be strictly contained in S_m.

The definition of strong alternative-excluding voting procedures depends on Condition U, a Pareto optimality condition.

CONDITION U. *Consider a strict committee* $\langle I_n, S_m, v^{nm}, T = T_p\rangle$. *The strict voting procedure* v^{nm} *satisfies Condition U if and only if, for every* $B = (B_1, ..., B_n) \in \rho_m{}^n$ *such that* $\Psi_T(B_1) = \Psi_T(B_2) \cdots \Psi_T(B_n)$, $v^{nm}(B) = \Psi_T(B_1)$.

Less formally, if v^{nm} satisfies Condition U and if the ballots unanimously rank $x \in T_p$ higher than every other $y \in T_p$, then v^{nm} will select x as the committee's choice. Given this, a strict voting procedure v^{nm} is a *strong alternative-excluding* voting procedure if and only if it is weak alternative-excluding and also satisfies Condition U.

Condition U is helpful in the proofs that follow because every strict strategy-proof voting procedure must satisfy it. Lemma 1 establishes this assertion.

LEMMA 1. *Consider a strict committee* $\langle I_n, S_m, v^{nm}, T = T_p\rangle$ *where* $n \geqslant 1, m \geqslant 3$, *and* $p \geqslant 1$. *If* v^{nm} *is strategy-proof, then it satisfies Condition U.*

Proof. Suppose v^{nm} is strategy-proof and does not satisfy Condition U. Consequently, for some $x \in T_p$ there exists a ballot set $C \in \rho_m{}^n$ such that $\Psi_T(C_1) = \Psi_T(C_2) = \cdots = \Psi_T(C_n)$ and $v^{nm}(C) = x \neq \Psi_T(C_1)$. Since

$\Psi_T(C_1) \in T_p$, a $D \in \rho_m{}^n$ exists such that $v^{nm}(D) = \Psi_T(C_1)$. Consider the sequence of ballot sets and outcomes:

$$\begin{aligned}
&v^{nm}(C_1, C_2, \ldots, C_n) = x \neq \Psi_T(C_1),\\
&v^{nm}(D_1, C_2, \ldots, C_n),\\
&\quad\vdots\\
&v^{nm}(D_1, \ldots, D_{i-1}, C_i, C_{i+1}, \ldots, C_n),\\
&v^{nm}(D_1, \ldots, D_{i-1}, D_i, C_{i+1}, \ldots, C_n),\\
&\quad\vdots\\
&v^{nm}(D_1, \ldots, D_{n-1}, C_n),\\
&v^{nm}(D_1, \ldots, D_{n-1}, D_n) = \Psi_T(C_1).
\end{aligned} \tag{3}$$

For later reference label such a sequence $S(C, D)$. At some point in this sequence of $n + 1$ elements the outcome must switch from $\sim\Psi_T(C_1)$ to $\Psi_T(C_1)$. Therefore, an $i \in I_n$ must exist such that

$$v^{nm}(D_1, \ldots, D_{i-1}, C_i, C_{i+1}, \ldots, C_n) = y \neq \Psi_T(C_1) \tag{4}$$

and

$$v^{nm}(D_1, \ldots, D_{i-1}, D_i, C_{i+1}, \ldots, C_n) = \Psi_T(C_1) \tag{5}$$

where $y \in T_p$ and $y \neq \Psi_T(C_1)$. Let individual i have preferences $R_i \equiv C_i$. This means that $\Psi_T(C_1)$ is that alternative contained within T_p which individual i most prefers. Consequently, his best strategy is the sophisticated strategy D_i rather than his sincere strategy C_i, i.e. v^{nm} is manipulable at $(D_1, \ldots, D_{i-1}, C_i, C_{i+1}, \ldots, C_n)$. Therefore, if v^{nm} fails to satisfy Condition U, then it is not strategy-proof. ∎

The next three lemmas prove that if a strict voting procedure $v^{n,3}$ defined for a three element alternative set is strategy-proof and has a range T_p, $1 \leqslant p \leqslant 3$, then it must be either fully dictatorial or strong alternative-excluding. The main task of these lemmas is to show that if $v^{n,3}$ is strategy-proof and $T_p = S_3$, then $v^{n,3}$ is fully dictatorial. The result that if $v^{n,3}$ is strategy proof and $T_p \Subset S_3$, then $v^{n,3}$ is strong alternative-excluding is secondary because it can be derived immediately. By definition $T_p \Subset S_3$ implies that $v^{n,3}$ is weak alternative-excluding. Since $v^{n,3}$ is both strategy-proof and weak alternative-excluding, Lemma 1 implies that $v^{n,3}$ is necessarily strong alternative-excluding.

The method of proof which the three lemmas together employ is mathematical induction over n, the number of individuals who are committee members. Lemma 2 begins the inductive chain by proving the result for committees with a single member.

LEMMA 2. *Consider a strict committee* $\langle I_1, S_3, v^{1,3}, T = T_p\rangle$ *where* $1 \leqslant p \leqslant 3$. *If* $v^{1,3}$ *is strategy-proof, then it is either fully dictatorial or strong alternative-excluding.*

Proof. Suppose the lemma is false. Therefore, a $v^{1,3}$ exists that is strategy-proof and neither fully dictatorial nor strong alternative-excluding. Then one of the following must be true: (a) $v^{1,3}$ satisfies Condition U and is not weak alternative-excluding, (b) $v^{1,3}$ satisfies Condition U and is weak alternative-excluding, or (c) $v^{1,3}$ does not satisfy Condition U. But Case (a) cannot be true since if $T_p = S_3$ and if $v^{1,3}$ satisfies Condition U, then $v^{1,3}$ must be fully dictatorial. This conclusion follows directly because for a single member committee Condition U is equivalent to a dictatoriality requirement. Case (b) cannot be true because any weak alternative-excluding voting procedure that satisfies Condition U is strong alternative-excluding. Case (c) also cannot be true because Lemma 1 states that every strategy-proof strict voting procedure satisfies Condition U. ∎

Statement and proof of Lemma 3 depends on the fact that we can write any strict voting procedure $v^{n,3}$ as an n-dimensional table. For example, let $(x\ y\ z)$ represent the ballot B_i with the properties that $x\ \bar{B}_i\ y$, $x\ \bar{B}_i z$, and $y\ \bar{B}_i z$ where $x, y, z \in S_3$. Tables I and II are then equivalent representations of an arbitrary, asymmetric strict voting procedure $v^{2,3}$. Specifically, if individuals one and two respectively cast ballots $(x\ z\ y)$ and $(y\ z\ x)$, then the committee's choice is z.

LEMMA 3. *Consider a strict committee* $\langle I_{n+1}, S_3, v^{n+1,3}, T_p\rangle$ *where* $n \geqslant 1$ *and* $1 \leqslant p \leqslant 3$. *Let* $B = (B_1, \ldots, B_n)$. *The strict voting procedure* $v^{n+1,3}$ *may be written as*

$$v^{n+1,3}(B, B_{n+1}) = \begin{cases} v_1^{n,3}(B) & \text{if } B_{n+1} = (x\ y\ z) \\ v_2^{n,3}(B) & \text{if } B_{n+1} = (x\ z\ y) \\ \quad\cdots & \\ v_6^{n,3}(B) & \text{if } B_{n+1} = (z\ y\ x) \end{cases} \tag{6}$$

where $v_1^{n,3}, \ldots, v_6^{n,3}$ *are strict voting procedures for committees with* n *members. No ballot set* $(B, B_{n+1}) \in \pi_m^{n+1}$ *exists at which any individual* i, *where* $i \in I_n$ *(individual* $n+1$ *is excluded), can manipulate* $v^{n+1,3}$ *if and only if each of the six voting procedures* $v_1{}^n, \ldots, v_6{}^n$ *are strategy-proof.*

Despite the if and only if phrasing, this lemma states that a necessary but not sufficient condition for constrcting a strategy-proof voting procedure $v^{n+1,3}$ is that it be constructed out of a set of strategy-proof voting proce-

TABLE I

$v^{2,3}(B_1, B_2)$

		B_1					
		$(x\ y\ z)$	$(x\ z\ y)$	$(y\ x\ z)$	$(y\ z\ x)$	$(z\ x\ y)$	$(z\ y\ x)$
B_2	$(x\ y\ z)$	x	x	y	y	y	y
	$(x\ z\ y)$	x	x	y	y	y	y
	$(y\ x\ z)$	y	y	x	x	x	x
	$(y\ z\ x)$	y	z	x	x	x	x
	$(z\ x\ y)$	y	y	x	x	x	x
	$(z\ y\ x)$	y	y	x	x	x	x

TABLE II

$v^{2,3}(B_1, B_2)$

$$v^{2,3}(B_1, B_2) = \begin{cases} v_1^{1,3}(B_1) \text{ if } B_2 = (x\ y\ z) \\ v_2^{1,3}(B_1) \text{ if } B_2 = (x\ z\ y) \\ v_3^{1,3}(B_1) \text{ if } B_2 = (y\ x\ z) \\ v_4^{1,3}(B_1) \text{ if } B_2 = (y\ z\ x) \\ v_5^{1,3}(B_1) \text{ if } B_2 = (z\ x\ y) \\ v_6^{1,3}(B_1) \text{ if } B_2 = (z\ y\ x) \end{cases}$$

Where

		$v_1^{1,3}$	$v_2^{1,3}$	$v_3^{1,3}$	$v_4^{1,3}$	$v_5^{1,3}$	$v_6^{1,3}$
B_1	$(x\ y\ z)$	x	x	y	y	y	y
	$(x\ z\ y)$	x	x	y	z	y	y
	$(y\ x\ z)$	y	y	x	x	x	x
	$(y\ z\ x)$	y	y	x	x	x	x
	$(z\ x\ y)$	y	y	x	x	x	x
	$(z\ y\ x)$	y	y	x	x	x	x

dures $v_k^{n,3}$, $k = 1,\ldots, 6$. The condition is not sufficient because some sets of voting procedures $v_k^{n,3}$ exist such that individual $n + 1$ can manipulate the resulting voting procedure $v^{n+1,3}$ in specific situations.

Proof. Suppose the necessary part is false. Therefore, a $v^{n+1,3}$ with its set of constituent $v_k^{n,3}$ must exist such that (a) $v^{n+1,3}$ is strategy proof for all individuals $j \in I_n$ and (b) some $v_k^{n,3}$, $1 \leqslant k \leqslant 6$, is not strategy proof for some individual $i \in I_n$. Without loss of generality suppose that $v_1^{n,3}$ is not strategy proof for individual i. Consequently there exists a ballot set $B = (B_1, ..., B_i, ..., B_n) \in \rho_3{}^n$ and ballot B_i' such that

$$v_1^{n,3}(B_1, ..., B_i', ..., B_n) \, \bar{B}_i v_1^{n,3}(B_1, ..., B_i, ..., B_n), \tag{7}$$

i.e., individual i can manipulate $v_1^{n,3}$ at B.

Let individual $n + 1$ cast ballot $B_{n+1} = (x\,y\,z)$. Let $B' = (B_1, ..., B_i', ..., B_n)$. This implies, based on (6), that

$$v^{n+1,3}(B, B_{n+1}) = v_1^{n,3}(B) \tag{8}$$

and

$$v^{n+1,3}(B', B_{n+1}) = v_1^{n,3}(B'). \tag{9}$$

Substitution into (7) gives

$$v^{n+1,3}(B', B_{n+1}) \, \bar{B}_i \, v^{n+1,3}(B, B_{n+1}) \tag{10}$$

which shows that $v^{n+1,3}$ is manipulable at (B, B_{n+1}). This contradicts the assumption that the lemma's necessary part is false.

Suppose the sufficient part is false. Therefore a $v^{n+1,3}$ with its set of constituent $v_1^{n,3}, ..., v_6^{n,3}$ must exist such that (a) $v_1^{n,3}, ..., v_6^{n,3}$ are strategy proof for all individuals $j \in I_n$ and (b) $v^{n+1,3}$ is not strategy proof for some individual $i \in I_n$. This implies that a ballot set $(B, B_{n+1}) = (B_1, ..., B_i, ..., B_n, B_{n+1}) \in \rho_3^{n+1}$ and ballot B_i' exist such that

$$v^{n+1,3}(B', B_{n+1}) \, \bar{B}_i v^{n+1,3}(B, B_{n+1}) \tag{11}$$

where $(B', B_{n+1}) = (B_1, ..., B_i', ..., B_n, B_{n+1})$. Assume without loss of generality that $B_{n+1} = (x\,y\,z)$. Equations (8) and (9) hold and therefore $v_1^{n,3}$ may be substituted for $v^{n+1,3}$:

$$v_1^{n,3}(B') \, \bar{B}_i v^{n,3}(B). \tag{12}$$

Thus, $v_1^{n,3}$ is not strategy-proof, a contradiction of the assumption that the sufficient part is false. ▌

Lemma 4 starts with the assumption that every strategy-proof strict voting procedure $v^{n,3}$ is either fully dictatorial or strong alternative-excluding. Then, with Lemma 3 as justification, it uses Eq. (6) and those voting procedures that we assume to be strategy-proof to construct every strategy-proof strict voting procedure $v^{n+1,3}$. The complication in this procedure is that a voting procedure $v^{n+1,3}$ is not necessarily strategy-proof if it is constructed out of strategy-proof voting procedures $v^{n,3}$. Depending on precisely how $v^{n+1,3}$ is constructed individual $n+1$ may find that in specific situations he can manipulate $v^{n+1,3}$.

LEMMA 4. *Consider a strict committee* $\langle I_{n+1}, S_3, v^{n+1,3}, T_p \rangle$ *where* $n \geqslant 1$ *and* $1 \leqslant p \leqslant 3$. *If every strategy-proof strict voting procedure* $v^{n,3}$ *is either fully dictatorial or strong alternative-excluding, then a necessary condition for* $v^{n+1,3}$ *to be strategy-proof is that it be either fully dictatorial or strong alternative-excluding.*

Proof. Let $\mathscr{V}^{n+1}$ be the collection of all strict voting procedures $v^{n+1,3}$ for committees with $n+1$ members. Let $\mathscr{X}^{n+1} \subset \mathscr{V}^{n+1}$ be the collection of all strict voting procedures $v^{n+1,3} \in \mathscr{V}^{n+1}$ that are fully dictatorial or strong alternative-excluding. Let $\mathscr{V}^n$ and $\mathscr{X}^n$ be the collections of strict voting procedures for committees with n members that correspond to $\mathscr{V}^{n+1}$ and $\mathscr{X}^{n+1}$ respectively. Let $\mathscr{W}^{n+1} \subset \mathscr{V}^{n+1}$ be the collection of all strict voting procedures $v^{n+1,3} \in \mathscr{V}^{n+1}$ that are constructed from voting procedures $v^{n,3} \in \mathscr{X}^n$, i.e., $v^{n+1,3} \in \mathscr{W}^{n+1}$ if and only if $v^{n+1,3}$ can be written as

$$v^{n+1,3}(B, B_{n+1}) = \begin{cases} v_1^{n,3}(B) & \text{if} \quad B_{n+1} = (x\,y\,z) \\ v_2^{n,3}(B) & \text{if} \quad B_{n+1} = (x\,z\,y) \\ \quad\cdots & \\ v_6^{n,3}(B) & \text{if} \quad B_{n+1} = (z\,y\,x) \end{cases} \tag{13}$$

where $B = (B_1, ..., B_n) \in \rho_m{}^n$ and $v_1^{n,3}, ..., v_6^{n,3} \in \mathscr{X}^n$. Finally let $\mathscr{V}^{n*}$ and $\mathscr{V}^{n+1*}$ be the collections of all strategy-proof strict voting procedures contained, respectively in the sets $\mathscr{V}^n$ and $\mathscr{V}^{n+1}$.

Assume that $\mathscr{V}^{n*} \subset \mathscr{X}^n$. Lemma 3 therefore implies $\mathscr{V}^{n+1*} \subset \mathscr{W}^{n+1}$. Consequently, every $v^{n+1,3} \in \mathscr{V}^{n+1*}$ can be identified by repeatedly partitioning $\mathscr{W}^{n+1}$ and discarding at every step those subsets which are disjoint with $\mathscr{V}^{n+1*}$. This partitioning of $\mathscr{W}^{n+1}$ depends on the fact that $\mathscr{X}^n$ contains seven classes of fully dictatorial and strong alternative-excluding voting procedures:

$$v^{n,3}(B) = f_T^i(B) \qquad \text{where} \quad T = S_3 \quad \text{and} \quad i \in I_n, \tag{14}$$

$$v^{n,3}(B) = h_K^{n,3}(B) = x, \tag{15}$$

$$v^{n,3}(B) = h_L^{n,3}(B) = y, \tag{16}$$

$$v^{n,3}(B) = h_M^{n,3}(B) = z, \tag{17}$$

$$v^{n,3}(B) = h_N^{n,3}(B), \tag{18}$$

$$v^{n,3}(B) = h_P^{n,3}(B), \qquad \text{and} \tag{19}$$

$$v^{n,3}(B) = h_Q^{n,3}(B), \tag{20}$$

where the notation $h_U^{n,3}$ represents a strong alternative-excluding voting procedure with range U and where $B \in \rho_m{}^n$, $S_3 = (x, y, z)$, $K = (x)$, $L = (y)$. $M = (z)$, $N = (y, z)$, $P = (x, z)$, and $Q = (x, y)$. Type (14) clearly represents every possible fully dictatorial voting procedure for a committee with n members. Types (15) through (20) exhaustively represent every possible strong alternative-excluding voting procedure because (K, L, M, N, P, Q) is the collection of all possible, proper, non-empty subsets of $S_3 = (x, y, z)$.

The set $\mathscr{W}^{n+1}$ can be partitioned into seven subsets:

$$\mathscr{W}_1^{n+1} = \{v^{n+1,3} \mid v^{n+1,3} \in \mathscr{W}^{n+1} \,\&\, v^{n+1,3}[B, (x\, y\, z)] = f_T^{\,i}(B) \text{ where } T = S_3 \text{ and } i \in I_n\}, \tag{21}$$

$$\mathscr{W}_2^{n+1} = \{v^{n+1,3} \mid v^{n+1,3} \in \mathscr{W}^{n+1} \,\&\, v^{n+1,3}[B, (x\, y\, z)] = h_K^{n,3}(B)\}, \tag{22}$$

$$\mathscr{W}_3^{n+1} = \{v^{n+1,3} \mid v^{n+1,3} \in \mathscr{W}^{n+1} \,\&\, v^{n+1,3}[B, (x\, y\, z)] = h_L^{n,3}(B)\}, \tag{23}$$

$$\ldots,$$

$$\mathscr{W}_7^{n+1} = \{v^{n+1,3} \mid v^{n+1,3} \in \mathscr{W}^{n+1} \,\&\, v^{n+1,3}[B, (x\, y\, z)] = h_Q^{n,3}(B)\}. \tag{24}$$

Each of these seven subsets can itself be partitioned into seven subsets: $\mathscr{W}_{11}^{n+1}, \ldots, \mathscr{W}_{17}^{n+1}, \mathscr{W}_{21}^{n+1}, \ldots, \mathscr{W}_{77}^{n+1}$.

Most of these subsets are easily proved to be disjoint with $\mathscr{V}^{n+1*}$. For example, consider

$$\mathscr{W}_{27}^{n+1} = \{v^{n+1,3} \mid v^{n+1,3} \in \mathscr{W}_2^{n+1} \,\&\, v^{n+1,3}[B, (x\, z\, y)] = h_Q^{n,3}(B)\}.$$

Let individual $n + 1$ have preferences and sincere strategy $R_{n+1} = (x\, z\, y)$ and let the other n individuals cast identical ballots $B_1 = B_2 = \cdots = B_n = (z\, y\, x)$. The definitions of $\mathscr{W}_{27}^{n+1}$, $h_Q^{n,3}$, and Condition U imply that $v^{n+1,3}[B, (x\, z\, y)] = h_Q^{n,3}(B) = y$. This is the least preferable outcome for individual $n + 1$. He can improve the outcome relative to his own preferences by employing the sophisticated strategy $B'_{n+1} = (x\, y\, z)$ because $v^{n+1,3}[B, (x\, y\, z)] = h_K^{n+1,3}(B) = x$. Therefore every $v^{n+1,3} \in \mathscr{W}_{27}^{n+1}$ is not strategy-proof. i.e., $\mathscr{W}_{27}^{n+1} \cap \mathscr{V}^{n+1*} = \varnothing$.

This procedure of elimination and partition may be continued through six levels until 17 subsets of $\mathscr{W}^{n+1}$ are identified that are not disjoint with $\mathscr{V}^{n+1*}$, i.e., these 17 subsets contain $\mathscr{V}^{n+1*}$. For example, one of these subsets $\mathscr{W}_{343344}^{n+1}$ contains a strategy-proof voting procedure of type $h_N^{n+1,3}$.

Inspection of these 17 subsets reveals that each one contains only strong alternative-excluding or fully dictatorial voting procedures. The specifics of this procedure are found in Satterthwaite [13]. Therefore $\mathscr{V}^{n+1*} = (\mathscr{V}^{n+1*} \cap \mathscr{W}^{n+1}) \subset \mathscr{X}^{n+1}$. ∎

Lemma 4 establishes an inductive chain on n whose initial assumption is validated by Lemma 2. Consequently Lemmas 2 and 4 together prove that if a strict voting procedure $v^{n,3}$ is strategy-proof, then it is either fully dictatorial or strong alternative-excluding. An inductive chain may also be established on m to generalize the results to any number of alternatives equal to or greater than three. I do not include the specifics of this step here because of their length; they may also be found in [13]. Lemma 5 summarizes this result.

LEMMA 5. *Consider a strict committee* $\langle I_n, S_m, v^{nm}, T_p\rangle$ *where* $n \geqslant 1$, $m \geqslant 3$ *and* $p \geqslant 1$. *If* v^{nm} *is strategy-proof, then it is either fully dictatorial or strong alternative-excluding.*

Two more steps are required to prove Theorem 1. Lemma 6 states that every strategy-proof strong alternative-excluding voting procedure must satisfy what is essentially an "independence of irrelevant alternatives" condition. The final step uses Lemma 6 to prove that every strategy-proof strong alternative-excluding voting procedure with a range of at least three alternatives must be partially dictatorial.

LEMMA 6. *Consider a strict committee* $\langle I_n, S_m, v^{nm}, T = T_p\rangle$ *where* $n \geqslant 2$, $m \geqslant 3$, $p \geqslant 1$, *and* $m \geqslant p$. *If* v^{nm} *is strategy-proof and two ballot sets* $C, D \in \rho_m{}^n$ *have the property that, for all* $i \in I_n$, $\theta_T(C_i) = \theta_T(D_i)$, *then* $v^{nm}(C) = v^{nm}(D)$.

The condition that $\theta_T(C_i) = \theta_T(D_i)$ for all $i \in I_n$ means that each pair of ballots—C_i and D_i—must have identical ordinal rankings of the elements contained within T_p.

Proof. If $T = S_m$, then the lemma is trivial because the condition placed on C and D implies that C must be identical to D. If $T \Subset S_m$, assume that v^{nm} is strategy-proof and, as a consequence of Lemma 1, strong alternative-excluding. Now suppose that this lemma is false. This means that a pair of ballot sets $C, D \in \rho_m{}^n$ exist such that (a) $v^{nm}(C) \neq v^{nm}(D)$ and (b), for all $i \in I_n$, $\theta_T(C_i) = \theta_T(D_i)$. Examine the sequence of ballot sets $S(C, D)$.[8] An $i \in I_n$ and distinct $x, y \in T$ must exist such that

$$v^{nm}(C_1, \ldots, C_{i-1}, D_i, D_{i+1}, \ldots, D_n) = x \tag{25}$$

[8] The sequence $S(C, D)$ is defined in lemma one's proof.

and

$$v^{nm}(C_1, ..., C_{i-1}, C_i, D_{i+1}, ..., D_n) = y. \tag{26}$$

Since we are considering strict committees indifference is ruled out. Therefore, because $\theta_T(C_i) = \theta_T(D_i)$, two cases are possible: either (*a*) $x\bar{C}_i y$ and $x\bar{D}_i y$ or (*b*) $y\bar{C}_i x$ and $y\bar{D}_i x$. If the former is true, then individual i can use D_i to manipulate v^{nm} at $(C_1, ..., C_{i-1}, C_i, D_{i+1}, ..., D_n)$. If the latter is true, individual i can use C_i to manipulate v^{nm} at $(C_1, ..., C_{i-1}, D_i, D_{i+1}, ..., D_n)$. Therefore, contrary to assumption, v^{nm} cannot be strategy-proof. ∎

This puts me in position to complete the proof of Theorem 1. It states that every strict voting procedure v^{nm} with a range of at least three elements is strategy-proof if and only if it is dictatorial. The if part is true by inspection. The only if part yields as follows. Lemma 5 states that if v^{nm} is strategy-proof, then it is either fully dictatorial or strong alternative-excluding Consequently, I need to show that if v^{nm} is strategy-proof and strong alternative-excluding then it is partially dictatorial. Assume that v^{nm} is strategy-proof, strong alternative-excluding and has a range $T = T_p$, $m > p \geqslant 3$.

For all $i \in I_n$ rewrite each ballot $B_i \in \rho_m{}^n$ as $B_i{}^* \in \rho_p{}^n$ where $B_i{}^*$ is a strong ordering, defined over T_p, with the property that $B_i{}^* = \theta_T(B_i)$. Each $B_i{}^*$ is identical to B_i except that the $m - p$ alternatives that are not included within the range of v^{nm} are deleted. Consider any $C \in \rho_m{}^n$ and $D \in \rho_m{}^n$, $C \neq D$, such that

$$[\theta_T(C_1), ..., \theta_T(C_n)] = [\theta_T(D_1), ..., \theta_T(D_n)]. \tag{27}$$

Lemma 6 implies that $v^{nm}(C) = v^{nm}(D)$. Consequently a strict voting procedure v^{np} for p alternatives exists such that, for all $B \in \rho_m{}^n$,

$$v^{np}[\theta_T(B_1), ..., \theta_T(B_n)] = v^{nm}(B_1, ..., B_n). \tag{28}$$

Since v^{nm} is strategy-proof, v^{np} is also strategy-proof and, by Lemma 5, is either dictatorial or strong alternative-excluding. It cannot be strong alternative-excluding because its range includes all p elements of T_p. Therefore it is dictatorial: an $i \in I_n$ exists that for all $B \in \rho_m{}^n$

$$v^{np}[\theta_T(B_1), ..., \theta_T(B_n)] = f_T{}^i[\theta_T(B_1), ..., \theta_T(B_n)]. \tag{29}$$

Substituting v^{nm} for v^{np} gives

$$v^{nm}(B_1, ..., B_n) = f_T{}^i[\theta_T(B_1), ..., \theta_T(B_n)] \tag{30}$$

$$= f_T{}^i(B_1, ..., B_n), \tag{31}$$

i.e., v^{nm} is partially dictatorial.

4. The Correspondence Theorem

In this section I show that the strategy-proofness condition for voting procedures corresponds precisely to Arrow's rationality, nonnegative response, citizens' sovereignty, and independence of irrelevant alternatives conditions for social welfare functions. Briefly the section's substance is as follows. Initially I restate Arrow's definitions of social welfare functions, rationality, nonnegative response (NNR), citizens' sovereignty (CS), and independence of irrelevant alternatives (IIA) and observe that every social welfare function is rational. Additionally I define strict social welfare functions as the analogue of strict voting procedures. Next I prove that a procedure exists for constructing a strict strategy-proof voting procedure from every strict social welfare function satisfying NNR, CS, and IIA. I then show that a procedure exists for constructing a strict social welfare function satisfying NNR, CS, and IIA from every strict strategy-proof voting procedure. This last result is based on an intermediate result which Gibbard [7] obtained in his proof of Theorem 1. Together these results imply the correspondence theorem: a one-to-one correspondence between strict strategy-proof voting procedures and strict social welfare functions satisfying NNR, CS, and IIA can be constructed. Section 6 contains the theorem's generalization to non-strict voting procedures and social welfare functions.

Arrow [1] defines a *social welfare function* for a committee with n members considering m alternatives to be a singlevalued mapping u^{nm} whose domain is $\pi_m{}^n$ and whose range is π_m or some nonempty subset of π_m. Thus $u^{nm}(B) = A$ where $B = (B_1, ..., B_n) \in \pi_m{}^n$ and $A \in \pi_m$. The weak order A is called the social ordering. A social welfare function is identical to a voting procedure except that its image is a weak order on S_m instead of a single element of S_m. Given a ballot set B and a subset $T \subset S_m$, Arrow defines the *social choice* over the set T to be $\Psi_T[u^{nm}(B)]$, i.e., the social choice is that element of T which the social ordering ranks highest. Finally, let a committee that is using a social welfare function u^{nm} be described by the triplet $\langle I_n, S_m, u^{nm}\rangle$.

Arrow's choice of definitions for social welfare function and social choice guarantees that every social welfare function satisfies the condition of rationality. Let $\eta_U(B)$ be a social choice function: for every $B \in \pi_m{}^n$ and $U \subset S_m$, the function's image is a subset of U. The function η is *rational* if for each $B \in \pi_m{}^n$ there exists a weak order $A \in \pi_m$ such that, for all $U \subset S_m$, $\eta_U(B) = \Psi_U(A)$. Thus, trivially, every social welfare function u^{nm} gives rise to rational social choices $\Psi_U[u^{nm}(B) = A]$.

In addition to the implicit requirement of rationality, Arrow posits four conditions which, he argues, any ideal social welfare function should satisfy.

Nondictatorship (ND). Let $A = u^{nm}(B)$. No $i \in I_n$ exists such that, for all $x, y \in S_m$ and for all $B \in \pi_m{}^n$, $x\bar{B}_i y$ implies $x\bar{A}y$.

Independence of Irrelevant Alternatives (IIA). Let $A_C = u^{nm}(C)$ and $A_D = u^{nm}(D)$. If for all $i \in I_n$, for some $W \subset S_m$, for some $C \in \pi_m{}^n$, and for some $D \in \pi_m{}^n$, $\theta_W(C_i), = \theta_W(D_i)$, then $\Psi_W(A_C) = \Psi_W(A_D)$.

Citizens' Sovereignty (CS). Let $A = u^{nm}(B)$. For every $x, y \in S_m$ there exists a ballot set $B \in \pi_m{}^n$ such that $x\bar{A}y$.

Nonnegative Response (NNR). For some $x \in S_m$ let $W = S_m - (x)$ and let $C, D \in \pi_m{}^n$ be any two ballot sets which have the properties that (a) for all $i \in I_n$, $\theta_W(C_i) = \theta_W(D_i)$, (b) for all $i \in I_n$ and all $y \in W$, $xD_i y$ if $xC_i y$, and (c) for all $i \in I_n$ and all $y \in W$, $x\bar{D}_i y$ if $x\bar{C}_i y$. Let $u^{nm}(C) = A_C$ and $u^{nm}(D) = A_D$. If, for any $z \in W$, $x\bar{A}_C z$, then $x\bar{A}_D z$.

In less formal language Condition NNR requires that if the only change in ballot set D is that on some individual ballots within ballot set D alternative x has been moved up relative to some other alternatives, then within the committee's final social ordering A_D alternative x cannot have moved down in relation to its position within the original social ordering A_C.

The reasonableness of the ND, CS, and NNR conditions is obvious. Fishburn [5] and Plott [12] contain excellent discussion of the reasonableness of rationality and IIA. Conditions CS, NNR, and IIA, as Arrow [1, pp. 97] has noted, imply the condition of Pareto optimality.

Pareto Optimality (PO). Let $A = u^{nm}(B)$. If any $B \in \pi_m{}^n$ has the property that $x\bar{B}_i y$ for all $i \in I_n$ and some $x, y \in S_m$, then $x\bar{A}y$.

Observe that if a social welfare function satisfiesPO, then it also satisfies CS.

I define a *strict social welfare function* analogously to a strict voting procedure. The domain of a strict social welfare function u^{nm} is limited to elements of $\rho_m{}^n$, i.e., only $B \in \rho_m{}^n$ are admissable as ballot sets. Similarly, the range of a strict social welfare function is limited; it may be either ρ_m or any of its nonempty subsets.

With these preliminaries complete, I can describe the procedure by which a strategy-proof voting procedure can be constructed from any social welfare function satisfying CS, NNR, and IIA. Let u^{nm} be a social welfare function with the property that, for all $B \in \pi_m{}^n$, the image of $\Psi_S[u^{nm}(B)]$ is always a single element of S_m. Construct the voting procedure v^{nm} by defining, for all $B \in \pi_m{}^n$, $v^{nm}(B) = \Psi_S[u^{nm}(B)]$. Call any v^{nm} so constructed the voting procedure *derived* from u^{nm}. Clearly the v^{nm} derived from a u^{nm} is unique. Lemma 7 states that a sufficient condition for a v^{nm} which is derived from a strict u^{nm} to be strategy-proof is that u^{nm} satisfy CS, NNR, and IIA.

LEMMA 7. *Consider a strict committee* $\langle I_n, S_m, u^{nm}\rangle$ *where* $n \geqslant 2$ *and* $m \geqslant 3$. *If the strict social welfare function satisfies* CS, NNR, *and* IIA, *then the strict voting procedure* v^{nm} *derived from* u^{nm} *is strategy-proof and has range* $T_v \equiv S_m$.

Proof. Since u^{nm} satisfies CS, NNR, and IIA, it also satisfies PO. Observe that if u^{nm} satisfies PO, then the derived v^{nm} has a range identical to S_m because u^{nm} has domain $\rho_m{}^n$ and $v^{nm}(B) = \Psi_S[u^{nm}(B)]$ for all $B \in \rho_m{}^n$. This leaves the question of the strategy-proofness of v^{nm}.

Suppose a strict u^{nm} satisfies CS, NNR, and IIA, and that its derived v^{nm} is not strategy-proof. Since v^{nm} is not strategy-proof a ballot set $(B_1, ..., B_i, ..., B_n) \in \rho_m{}^n$ exists at which v^{nm} is manipulable:

$$v^{nm}(B_1, ..., B_i', ..., B_n)\, \bar{B}_i v^{nm}(B_1, ..., B_i, ..., B_n) \tag{32}$$

where $B_i' \in \rho_m$. Let $v^{nm}(B_1, ..., B_i', ..., B_n) = x$, $v^{nm}(B_1, ..., B_i, ..., B_n) = y$, $u^{nm}(B_1, ..., B_i', ..., B_n) = A'$, and $u^{nm}(B_1, ..., B_i, ..., B_n) = A$ where A, $A' \in \rho_m$. Note that by definition $\Psi_S(A') = x$ and $\Psi_S(A) = y$. Consequently, (32) may be rewritten as $\Psi_S(A')\, \bar{B}_i \Psi_S(A)$ or as $x\bar{B}_i y$. Focusing now on B_i', two possibilities exist: $y\bar{B}_i'x$ or $x\bar{B}_i'y$.

Consider the first case where $y\bar{B}_i'x$. Let $U = S_m - (x)$. Construct a new ballot $B_i^* = [x\ \theta_U(B_i')]$, i.e., $x\bar{B}_i^*z$ for all $z \in U$ and, for all w, $z \in U$, $w\bar{B}_i^*z$ if and only if $w\bar{B}_i'z$. Thus, on the ballot B_i^* alternative x is top-ranked and the relative positions of other alternatives is unchanged. This is the type of shift that condition NNR describes. Let $u^{nm}(B_1, ..., B_i^*, ..., B_n) = A^*$. Nonnegative response (NNR) then implies that $x\bar{A}^*z$ for all $z \in U$. This is because $\Psi_S(A') = x$; consequently, $\Psi_S(A^*) = x$ also. Let $X = (x, y)$. Notice B_i^* is constructed so that $\theta_X(B_i^*) = \theta_X(B_i)$, i.e., both $x\bar{B}_i^*y$ and $x\bar{B}_i y$. If we apply IIA, the implication is that $\Psi_X(A^*) = \Psi_X(A)$. This, however, contradicts the assumption that $\Psi_X(A^*) = \Psi_S(A^*) = x$ and $\Psi_X(A) = \Psi_S(A) = y$. Therefore, if $y\bar{B}_i'x$, then v^{nm} must be strategy-proof.

Consider the second case where $x\bar{B}_i'y$. Observe that $\theta_X(B_i) = \theta_X(B_i')$ where $X = (x, y)$. Condition IIA implies that necessarily $\Psi_X(A') = \Psi_X(A)$. This, however, contradicts the assumption that $\Psi_X(A) = \Psi_S(A) = y$ and $\Psi_X(A') = \Psi_S(A') = x$. Therefore, if $x\bar{B}_i'y$, then v^{nm} must be strategy-proof. ∎

Consistent with the definition of a derived voting procedure, I define u^{nm} to be the social welfare function that *underlies* the voting procedure v^{nm} if and only if, for all $B \in \pi_m{}^n$, $\Psi_S[u^{nm}(B)] = v^{nm}(B)$ where $S = S_m$. Clearly, many social welfare functions underlie every voting procedure v^{nm}. My interest here, however, is to find for each strategy-proof voting procedure v^{nm} an underlying social welfare function u^{nm} that satisfies CS, NNR, and IIA. Such a u^{nm} can be constructed for any strict strategy-proof v^{nm} by following a procedure Gibbard [7] has devised.

Pick an arbitrary strong order $Q \in \rho_m$. Define Δ_{xy}, where $x, y \in S_m$ and $x \neq y$, to be a function with domain and range ρ_m. Let Δ_{xy} have properties such that if $B_i^* = \Delta_{xy}(B_i)$, then

(a) $x\bar{B}_i^*y$ if $x\bar{B}_iy$, $y\bar{B}_i^*x$ if $y\bar{B}_ix$,

(b) $x\bar{B}_i^*w$ and $y\bar{B}_i^*w$ for all $w \in S_m - (x, y)$, and

(c) $w\bar{B}_i^*z$ if wQz for all $w, z \in S_m - (x, y)$.

For each ballot set $(B_1, ..., B_n)$ construct a binary relation P such that, for all $x, y \in S_m$ and $x \neq y$, $x\bar{P}y$ if and only if $x = v^{nm}[\Delta_{xy}(B_1), ..., \Delta_{xy}(B_n)]$. Since a P is defined for each ballot set $(B_1, ..., B_n) \in \rho_m{}^n$, a function μ can be defined that associates the appropriate P with each $B \in \rho_m{}^n$, Gibbard [7], in his proof of this paper's Theorem 1, showed that if a strict voting procedure v^{nm} is strategy-proof, then the binary relation P associated with each $B \in \rho_m{}^n$ is a strong order, i.e. $P \in \rho_m$. This means that the function μ is a strict social welfare function. Gibbard then went on to show that μ has two properties in such cases: it underlies v^{nm} and it satisfies PO and IIA.[9] It also satisfies CS because PO implies CS.

Three facts are important to note concerning Gibbard's result. First, it considers only strict voting procedures whose ranges T_p are identical to the alternative set S_m. Nevertheless, this restriction is not limiting because, as was shown in section three's proof of Theorem 1, any strict strategy-proof v^{nm} for which $T_p \Subset S_m$, $p \geqslant 3$, can be rewritten as a strict strategy-proof voting procedure v^{np} defined over the reduced alternative set $S_p \equiv T_p$. Gibbard's result then applies to v^{np}: underlying it is a strict social welfare function u^{np} which satisfies PO and IIA.

The second fact to note is that Gibbard's result does not establish the uniqueness of the u^{nm} that underlies each strict strategy-proof v^{nm}. This, however, is easy to prove. Suppose that two strict social welfare functions μ and μ' both underlie v^{nm}, both satisfy PO and IIA, and, for some $C \in \rho_m{}^n$, $\mu(C) \neq \mu'(C)$. Observe that, for all $B \in \rho_m{}^n$, $v^{nm}(B) = \Psi_S[\mu(B)] = \Psi_S[\mu'(B)]$ because μ and μ' are both assumed to underlie v^{nm}. Therefore, an $x, y \in S_m$ exist such that $x\bar{A}y$ and $y\bar{A}'x$ where $\mu(C) = A$ and $\mu'(C) = A'$. Let, for all $i \in I_n$, $C_i^* = \Delta_{xy}(C_i)$. Also let $A^* = \mu(C^*)$ and $A'^* = \mu'(C^*)$. By IIA, $x\bar{A}^*y$ and $y\bar{A}'^*x$. By PO, $x\bar{A}^*z$, $y\bar{A}^*z$, $x\bar{A}'^*z$, $y\bar{A}'^*z$ for all

[9] This particular result is the heart of Gibbard's proof of this paper's Theorem 1. His method is to first show that underlying every strategy-proof v^{nm} is a u^{nm} satisfying PO and IIA. Arrow's general possibility theorem [1] then implies that u^{nm} is dictatorial. Finally, he proves that a dictatorial u^{nm} underlying v^{nm} implies that v^{nm} is dictatorial.

$z \in S_m - (x, y)$. Therefore, $\Psi_S(A^*) = \Psi_S[\mu(C^*)] = x$ and $\Psi_S(A'^*) = \Psi_S[\mu'(C^*)] = y$. This contradicts our original assumption that, for all $B \in \rho_m{}^n$, $\Psi_S[\mu(B)] = \Psi_S[\mu'(B)] = v^{nm}(B)$. Consequently $\mu = \mu'$, i.e., only one social welfare function satisfying PO and IIA underlies each strict strategy-proof voting procedure.

Third, note that Gibbard's result does not assert that the u^{nm} underlying a strategy-proof v^{nm} satisfies NNR. Suppose u^{nm} satisfies PO and IIA, but does not satisfy NNR. Consequently, $x, y \in T_p$, $B = (B_1, ..., B_i, ..., B_n) \in \rho_m{}^n$, and $B_i' \in \rho_m$ exist such that $y\bar{B}_i x$, $x\bar{B}_i' y$, $x\bar{A}y$, and $y\bar{A}'x$ where $A = u^{nm}(B_1, ..., B_i, ..., B_n)$, and $A' = u^{nm}(B_1, ..., B_i', ..., B_n)$. Let, for all $j \in I_n$, $C_j = \Delta_{xy}(B_j)$ and $C_i' = \Delta_{xy}(B_i')$. Therefore, because u^{nm} satisfies IIA and PO, $\Psi_S[u^{nm}(C_1, ..., C_i, ..., C_n)] = x$ and $\Psi_S[u^{nm}(C_1, ..., C_i', ..., C_n)] = y$. Recall that, since u^{nm} underlies v^{nm}, $\Psi_S[u^{nm}(B)] = v^{nm}(B)$. Therefore, $v^{nm}(C_1, ..., C_i, ..., C_n) = x$ and $v^{nm}(C_1, ..., C_i', ..., C_n) = y$. Because $y\bar{C}_i x$ individual i can manipulate v^{nm} at $(C_1, ..., C_i, ..., C_n)$. Therefore, since v^{nm} is not strategy-proof if u^{nm} violates NNR, u^{nm} must necessarily satisfy NNR. Lemma 8 summarizes these results. Lemmas 7 and 8 together prove Theorem 2.

LEMMA 8. *Consider a strict committee* $\langle I_n, S_m, v^{nm}, T_p \rangle$ *where* $n \geqslant 2$, $m \geqslant 3$, *and* $T_p \equiv S_m$. *If* v^{nm} *is strategy-proof, then there exists a unique, strict social welfare function* u^{nm} *which underlies* v^{nm} *and satisfies* CS, NNR, *and* IIA.

THEOREM 2. *Let* $n \geqslant 2$ *and* $m \geqslant 3$. *A one-to-one correspondence* λ *exists between every strict strategy-proof voting procedure* v^{nm} *with range* $T_p \equiv S_m$ *and every strict social welfare function* u^{nm} *satisfying* CS, NNR, *and* IIA. *If* $u^{nm} = \lambda(v^{nm})$, *then* u^{nm} *underlies* v^{nm} *and* v^{nm} *is derived from* u^{nm}.

This theorem's significance stems from the fact that the strategy-proofness condition corresponds to Arrow's rationality, CS, NNR, and IIA conditions independently of the fact that each set of conditions by itself implies dictatoriality. Thus, construction of a social welfare function satisfying Arrow's conditions is equivalent to constructing a strategy-proof voting procedure.

Theorem 2 creates a strong new justification for Arrow's choice of rationality, CS, NNR, and IIA as conditions which an ideal social welfare function should satisfy. The conditions of rationality and IIA which have caused so much controversy are now shown to be part and parcel of the very practical criterion of strategy-proofness. For instance, this theorem shows that rationality is more than an attractive intellectural criterion. If a social welfare function violates rationality, then the voting procedure derived from it violates strategy-proofness.

5. Arrow's General Possibility Theorem

If a strict social welfare satisfies CS, NNR, and IIA, then Lemma 7 states that the strict voting procedure derived from it is strategy-proof. According to Theorem 1 this derived voting procedure must be dictatorial. In this section I show that dictatoriality of the derived voting procedure implies dictatoriality of the original social welfare function. This establishes for the case of strict social welfare functions a new proof of Arrow's general possibility theorem [1]. In Section 6 I extend this proof to the general case of social welfare functions.

THEOREM 3. (Arrow). *Consider a strict committee* $\langle I_n, S_m, u^{nm}\rangle$ *where* $n \geqslant 2$ *and* $m \geqslant 3$. *The strict social welfare function* u^{nm} *satisfies* CS, NNR, *and* IIA *if and only if it is dictatorial.*

Proof. Suppose a strict u^{nm} exists which is not dictatorial, but which satisfies CS, NNR, and IIA. By Lemma 7, let v^{nm} be the strategy-proof strict voting procedure derived from u^{nm}. By the constructive proof of Theorem 2, v^{nm} is dictatorial. Hence, for all $B \in \rho_m{}^n$, $\Psi_S[u^{nm}(B)] = v^{nm}(B) = f_S{}^i(B)$ for some $i \in I_n$ and where $S = S_m$. Recall, however, that u^{nm} is not dictatorial. This implies that a ballot set $B \in \rho_m{}^n$ exists such that, for some $x, y \in S_m$, $x\bar{B}_i y$ and $y\bar{A}x$ where $u^{nm}(B) = A$.

Rewrite ballot set B as B^* where, for all $j \in I_n$, $B_j{}^* = [\theta_U(B_j)\ \theta_W(B_j)]$, $U = (x, y)$, and $W = S_m - (x, y)$, i.e. $B_j{}^*$ is identical to B_j except that alternatives x and y are moved to the top. Consequently $f_S{}^i(B^*) = x$ because $x\bar{B}_i y$ implies $x\bar{B}_i{}^*y$. Let $A^* = u^{nm}(B^*)$. By IIA, $y\bar{A}^*x$. By PO, either $\Psi_S(A^*) = x$ or $\Psi_S(A^*) = y$. The former is impossible because $y\bar{A}^*x$. Therefore, $\Psi_S[u^{nm}(B^*)] = v^{nm}(B^*) = y$. This, however, contradicts the fact that individual i is a dictator for v^{nm} because $v^{nm}(B^*) = f_S{}^i(B^*) = x$. Therefore u^{nm} cannot be nondictatorial. ▮

6. Generalizations to Weak Orders

In this final section I generalize Theorems 1, 2, and 3 by making indifference admissable on individuals' ballots and preferences. The key step in my proofs of these generalization is to show that strategy-proof voting procedures and social welfare functions satisfying CS, NNR, and IIA may be decomposed into a tie-breaking function and, respectively, a strict voting procedure or strict social welfare function.

Define a *tie-breaking function* to be a single-valued function α with domain $\pi_m{}^n$, range $\rho_m{}^n$, and property that if $\alpha(B) = C$ for some $B \in \pi_m{}^n$ and $C \in \rho_m{}^n$, then, for all $x, y \in S_m$, and all $i \in I_n$, $x\bar{B}_i y$ implies $x\bar{C}_i y$ and $y\bar{B}_i x$ implies $y\bar{C}_i x$. If, for some $x, y \in S_m$ and some $B_i \in \pi_m$, $xB_i y$ and $yB_i x$ then either $x\bar{C}_i y$ or $y\bar{C}_i x$ depending on the tie-breaking function's structure. Every tie-breaking function α decomposes into n component tie-breaking functions: $\alpha[B] = [\alpha_1(B),..., \alpha_i(B),..., \alpha_n(B)] = [C_1,..., C_i,..., C_n]$. A *regular tie-breaking function* γ is a tie-breaking function for which a set of strong orders $Q = (Q_1,..., Q_i,..., Q_n) \in \rho_m{}^n$ exists such that if $C = \gamma(B)$ and, for some $x, y \in S_m$, $xB_i y$ and $yB_i x$, then $x\bar{C}_i y$ if and only if $x\bar{Q}_i y$. Any regular tie-breaking function γ decomposes into n independent component tie-breaking functions: $\gamma(B) = [\gamma_1(B_1),..., \gamma_i(B_i),..., \gamma_n(B_n)] = [C_1,..., C_i,..., C_n]$. Call the ordering Q_i the tie-breaking order for the component function γ_i.

Table III defines two illustrative component tie-breaking functions, α_i and γ_i, which have as their arguments only the ballot B_i instead of the entire ballot set B. Let the notation $B_i = (x \approx y\ z)$ represent a ballot

TABLE III

Tie-breaking Functions α and γ.

B_i	$\alpha_i(B_i)$	$\gamma_i(B_i)$
$(x\ y\ z)$	$(x\ y\ z)$	$(x\ y\ z)$
$(x\ z\ y)$	$(x\ z\ y)$	$(x\ z\ y)$
$(y\ x\ z)$	$(y\ x\ z)$	$(y\ x\ z)$
$(y\ z\ x)$	$(y\ z\ x)$	$(y\ z\ x)$
$(z\ x\ y)$	$(z\ x\ y)$	$(z\ x\ y)$
$(z\ y\ x)$	$(z\ y\ x)$	$(z\ y\ x)$
$(x \approx y \approx z)$	$(x\ y\ z)$	$(x\ y\ z)$
$(x \approx y\ z)$	$(y\ x\ z)$	$(x\ y\ z)$
$(x\ y \approx z)$	$(x\ z\ y)$	$(x\ y\ z)$
$(x \approx z\ y)$	$(z\ x\ y)$	$(x\ z\ y)$
$(y\ x \approx z)$	$(y\ x\ z)$	$(y\ x\ z)$
$(y \approx z\ x)$	$(y\ z\ x)$	$(y\ z\ x)$
$(z\ x \approx y)$	$(z\ y\ x)$	$(z\ x\ y)$

Key: $B_i = (x\ y\ z)$ means $x\bar{B}_i y$, $x\bar{B}_i z$, and $y\bar{B}_i z$.
$B_i = (x\ y \approx z)$ means $x\bar{B}_i y$, $x\bar{B}_i z$, $yB_i z$, and $zB_i y$.

$B_i \in \pi_3$ such that xB_iy, yB_ix, $x\bar{B}_iz$, and $y\bar{B}_iz$. The functions α_i and γ_i break the indifference between x and y in opposite directions: $\alpha_i(x \approx yz) = (y\,x\,z)$ and $\gamma_i(x \approx y\,z) = (x\,y\,z)$. Function γ_i on Table III is admissible as a component of a regular tie-breaking function because the strong ordering $Q_i = (x\,y\,z)$ describes how γ_i breaks indifference between the elements of S_3. Function α_i, however, is not admissible as a component of a regular tie-breaking function because, for instance, it breaks indifference between y and z in both directions: $\alpha_i(x\,y \approx z) = (x\,z\,y)$ and $\alpha_i(y \approx z\,x) = (y\,z\,x)$.

Based on this definition of regular tie-breaking functions, I define a *regular voting procedure* to be any voting procedure v^{nm} which can be written such that, for all $B \in \pi_m{}^n$,

$$v^{nm}(B) = \nu^{nm}[\gamma(B)] \tag{33}$$

where ν^{nm} is a strict voting procedure and γ is a regular tie-breaking function. Similarly, I define a *regular social welfare* function to be any social welfare function u^{nm} whose range is contained in ρ_m and which can be written such that, for all $B \in \pi_m{}^n$,

$$u^{nm}(B) = \mu^{nm}[\gamma(B)] \tag{34}$$

where μ^{nm} is a strict social welfare function and γ is a regular tie-breaking function. Notice that the range of a regular social welfare function is limited to ρ_m instead of π_m.

With these definitions in hand I shall in the remainder of this section state and prove Theorems 1′, 2′, and 3′. These theorems generalize Theorems 1, 2, and 3 from strict to nonstrict committees. In their proofs I shall use the results from three additional lemmas that I also state and prove in this section. These lemmas, which have interest in their own right, show how strategy-proof voting procedures and social welfare functions satisfying CS, NNR, and IIA can be decomposed into tie-breaking functions and, respectively, strict strategy-proof voting procedures and strict social welfare functions satisfying PO and IIA.

LEMMA 9. *Consider a committee $\langle I_n, S_m, v^{nm}, T_p\rangle$. If, for all $B \in \pi_m{}^n$, $v^{nm}(B) = v^{nm}[\gamma(B)]$ where ν^{nm} is a strict strategy-proof voting procedure and γ is a regular tie-breaking function, then v^{nm} is strategy-proof. If v^{nm} is strategy proof, then there exists a strict strategy-proof voting procedure ν^{nm} and tie-breaking function α such that, for all $B \in \pi_m{}^n$, $v^{nm}(B) = \nu^{nm}[\alpha(B)]$.*

Proof. Suppose a strict strategy-proof ν^{nm} and regular tie-breaking function $\gamma(B) = [\gamma_1(B_1),..., \gamma_n(B_n)]$ exist such that the voting procedure

$v^{nm}(B) = \nu^{nm}[\gamma(B)]$ is not strategy proof. Therefore, a $B \in \pi^{nm}$ exists at which v^{nm} is manipulable:

$$v^{nm}(B_1, ..., B_j', ..., B_n)\, \bar{B}_j v^{nm}(B_1, ..., B_j, ..., B_n) \tag{35}$$

where $B_j' \in \pi_m$. Let $C = \gamma(B)$. Since v^{nm} is assumed decomposable: $v_{nm}(B_1, ..., B_j', ..., B_n) = \nu^{nm}[\gamma_1(B_1), ..., \gamma_i(B_j'), ..., \gamma_n(B_n)] = \nu(C_1, ..., C_j', ..., C_n) = x$ and $v^{nm}(B_1, ..., B_j, ..., B_n) = \nu^{nm}[\gamma_1(B_1), ..., \gamma_i(B_j), ..., \gamma_n(B_n)] = \nu^{nm}(C_1, ..., C_j, ..., C_n) = y$ where $x, y \in S_m$. Relationship (35) implies $x\bar{B}_j y$ which in turn implies $x\bar{C}_j y$. This allows us to substitute ν^{nm} for v^{nm} in (35):

$$\nu^{nm}(C_1, ..., C_j', ..., C_n)\, \bar{C}_j \nu^{nm}(C_1, ..., C_j, ..., C_n), \tag{36}$$

i.e., ν^{nm} is manipulable at $(C_1, ..., C_j, ..., C_n)$. Consequently, our assumption that ν^{nm} is strategy-proof is contradicted.

Consider the lemma's second proposition now. I start with a strategy-proof v^{nm} and must show that there exists a strict strategy-proof ν^{nm} and tie-breaking function α such that, for all $B \in \pi_m{}^n$, $v^{nm}(B) = \nu^{nm}[\alpha(B)]$. First, I define the strict voting procedure ν^{nm} such that $\nu^{nm}(B_1, ..., B_n) = v^{nm}(B_1, ..., B_n)$ for all $B \in \rho_m{}^n$. This definition guarantees the strategy-proofness of ν^{nm} because v^{nm}, by virtue of its strategy-proofness over its domain $\pi_m{}^n$, cannot be manipulated at any point in the domain $\rho_m{}^n$ of ν^{nm}.

To complete the proof I must construct tie-breaking functions $\alpha = (\alpha_1, ..., \alpha_n)$. An iterative process of first finding an appropriate α_1, then an appropriate α_2, and so on through α_n works. Consider an arbitrary ballot set $B \in \pi_m{}^n$ and suppose I have found, for some $j \in I_n$, appropriate α_i for all $i < j$, i.e.,

$$v^{nm}(B_1, ..., B_j, ..., B_n) = v^{nm}(\alpha_1(B), ..., \alpha_{j-1}(B), B_j, ..., B_n) = x. \tag{37}$$

Further suppose I cannot find an appropriate tie-breaker α_j, i.e., for every α_j

$$v^{nm}(\alpha_1(B), ..., \alpha_{j-1}(B), \alpha_j(B), B_{j+1}, ..., B_n) = y, \tag{38}$$

where $y \neq x$. Pick any α_j such that (38) is true. Let $C_i = \alpha_i(B)$ for all $i \leqslant j$. The assumption that v^{nm} is strategy-proof implies two conditions:

$$\sim v^{nm}(C_1, ..., C_{j-1}, B_j, B_{j+1}, ..., B_n)\, \bar{C}_j v^{nm}(C_1, ..., C_{j-1}, C_j, B_{j+1}, ..., B_n)$$

and (39)

$$\sim v^{nm}(C_1, ..., C_{j-1}, C_j, B_{j+1}, ..., B_n)\, \bar{B}_j v^{nm}(C_1, ..., C_{j-1}, B_j, B_{j+1}, ..., B_n). \tag{40}$$

These may be rewritten, based on (37) and (38), as $\sim x\bar{C}_j y$ and $\sim y\bar{B}_j x$. Since $C_j = \alpha_j(B)$, $x\bar{B}_j y$ would imply $x\bar{C}_j y$. Nevertheless, $\sim x\bar{C}_j y$; therefore $\sim x\bar{B}_j y$. Together $\sim x\bar{B}_j y$ and $\sim y\bar{B}_j x$ indicate indifference between x and y on ballot B_j. Moreover, since $C_j \in \rho_m$, $\sim x\bar{C}_j y$ implies $y\bar{C}_j x$. In summary, strategy-proofness of v^{nm} implies $xB_j y$, $yB_j x$, and $y\bar{C}_j x$.

The conclusion is clear: if, for a strategy-proof v^{nm}, breaking the tie on ballot B_j changes the committee's choice from x to y, then necessarily the ballot B_j ranks x and y indifferently and the tie-breaker α_j moves y above x. This conclusion, however, contradicts the assumption that no appropriate α_j exists. Let α_j' break the tie between x and y in favor of x instead of in favor of y. The conclusion stated above implies that no change in the committee's choice can result because α_j' breaks the indifference in favor of the committee's original choice. Therefore α_j' is an appropriate α_j. Since my original choices of both j and $B = (B_1, \ldots, B_n)$ were arbitrary, I can find an appropriate $\alpha_j(B)$ for each $j \in I_n$ and each $B \in \pi_m{}^n$. ▌

THEOREM 1′. *Consider a committee* $\langle I_n, S_m, v^{nm}, T_p\rangle$ *where* $n \geqslant 2$ *and* $m \geqslant p \geqslant 3$. *The voting procedure* v^{nm} *is strategy-proof only if it is dictatorial.*

Proof. The proof follows from Lemma 9 which states that since v^{nm} is strategy-proof it can be written $v^{nm}(B) = \nu^{nm}[\alpha(B)]$ where ν^{nm} is a strict strategy-proof voting procedure and α is a tie-breaking function. By Theorem 1, $\nu(C) = f_T{}^i(C) = \Psi_T(C_i)$ for some $i \in I_n$ and all $C \in \rho_m{}^n$. Let $C_i = \alpha_i(B)$. Therefore, for all $B \in \pi_m{}^n$, $v^{nm}(B) = \Psi_T[\alpha_i(B)] = f_T{}^i(B)$ because $f_T{}^i$ implicitly incorporates the component tie-breaking function α_i. ▌

LEMMA 10. *Consider a committee* $\langle I_n, S_m, u^{nm}\rangle$ *where* $n \geqslant 2$, $m \geqslant 3$, *and* u^{nm} *is a social welfare function with domain* $\pi_m{}^n$ *and range contained in* ρ_m. *If, for all* $B \in \pi_m{}^n$, $u^{nm}(B) = \mu^{nm}[\gamma(B)]$ *where* γ *is a regular tie-breaking function and* μ *is a strict social welfare function satisfying* IIA, CS, *and* NNR, *then* u^{nm} *satisfies* IIA, CS, *and* NNR. *If* u^{nm} *satisfies* IIA, CS, *and* NNR, *then there exists a tie-breaking function* α *and a strict social welfare function* μ^{nm} *satisfying* IIA, CS, *and* NNR *such that, for all* $B \in \pi_m{}^n$, $u^{nm}(B) = \mu^{nm}[\alpha(B)]$.

Proof. Suppose that $\gamma = (\gamma_1, \ldots, \gamma_n)$ is a regular tie-breaking function and μ^{nm} satisfies CS, NNR, and IIA. Let $u^{nm}(B) = \mu^{nm}[\gamma(B)]$. Obviously, since μ^{nm} satisfies CS and, by definition, u^{nm} and μ^{nm} have identical ranges, u^{nm} satisfies CS. Suppose, however, that u^{nm} does not satisfy IIA. Consequently, there must exist a $B \in \pi_m{}^n$, a $C \in \pi_m{}^n$, and a $U \Subset S_m$ such that

$[\theta_U(B_1),..., \theta_U(B_n)] = [\theta_U(C_1),..., \theta_U(C_n)]$ and $\Psi_U(A_B) \neq \Psi_U(A_C)$ where $u^{nm}(B) = A_B$ and $u^{nm}(C) = A_C$.

Let $B' = (B_1',..., B_n') = [\gamma_1(B_1),..., \gamma_n(B_n)]$, $C' = (C_1',..., C_n') = [\gamma_1(C_1),..., \gamma_n(C_n)]$, $\mu^{nm}(B') = A_B'$, and $\mu^{nm}(C') = A_C'$. Observe that the definition of u^{nm} implies that $A_B = A_B'$ and $A_C = A_C'$. Also observe that $[\theta_U(B_1'),..., \theta_U(B_n')] = [\theta_U(C_1'),..., \theta_U(C_n')]$ because the definition of regular tie-breakers guarantees that, for all $i \in I_n$, $\theta_U[\gamma_i(B_i)] = \theta_U[\gamma_i(C_i)]$ if $\theta_U(B_i) = \theta_U(C_i)$. Since μ^{nm} satisfies IIA, these last two results mean that $\Psi_U(A_B') = \Psi_U(A_C')$ and $\Psi_U(A_B) = \Psi_U(A_C)$. But this contradicts the assumption that $\Psi_U(A_B) \neq \Psi_U(A_C)$. Therefore, u^{nm} must satisfy IIA. A similar argument can be constructed to show that u^{nm} must satisfy NNR.

The proof of the lemma's second proposition parallels the proof of the second proposition of Lemma 9. Assume u^{nm} satisfies IIA, NNR, and CS. Define the strict social welfare function μ^{nm} such that $\mu^{nm}(B) = u^{nm}(B)$ for all $B \in \rho_m{}^n$. Obviously μ^{nm} satisfies NNR, CS, and IIA. Consider an arbitrary $B \in \pi_m{}^n$ and suppose, for some $j \in I_n$ and all $i < j$, appropriate α_i exist:

$$u^{nm}(B_1 ,..., B_{j-1} ,..., B_n) = u^{nm}[\alpha_1(B),..., \alpha_{j-1}(B), B_j ,..., B_n] = A \quad (41)$$

where $A \in \rho_m$. Assume an appropriate tie-breaker α_j does not exist, i.e., for all α_j

$$u^{nm}[\alpha_1(B),..., \alpha_{j-1}(B), \alpha_j(B), B_{j+1} ,..., B_n) = A' \quad (42)$$

where $A' \in \rho_m$ and $A' \neq A$. Pick any α_j such that (42) is true. Therefore, some $x, y \in S_m$ exist such that $x\bar{A}y$ and $y\bar{A}'x$. Let $B_j' = \alpha_j(B)$. Observe that IIA implies that the difference in how A and A' rank x and y must stem from a difference in how B_j and B_j' rank x and y. Two conclusions follow from this observation, NNR, and the definition of α_j: (a) xB_jy and yB_jx and (b) $y\bar{B}_j'x$. Define $\alpha_j{}^*$ such that $x\bar{B}_j{}^*y$ where $B_j{}^* = \alpha_j{}^*(B)$. NNR in conjunction with $x\bar{A}y$ implies that $x\bar{A}^*y$ where $A^* = u^{nm}(...\alpha_{j-1}(B), \alpha_j{}^*(B), B_{j+1} ,...)$. Thus $\alpha_j{}^*$ is a component tie-breaking function that works. ▮

THEOREM 2'. *Let $n \geqslant 2$ and $m \geqslant 3$. A one-to-one correspondence λ exists between every regular strategy-proof voting procedure v^{nm} and every regular social welfare function satisfying* CS, NNR, *and* IIA. *If $u^{nm} = \lambda(v^{nm})$, then u^{nm} uniquely underlies v^{nm} and v^{nm} is uniquely derived from u^{nm}.*

This theorem generalizes Theorem 2 only to regular strategy-proof voting procedures and regular social welfare functions satisfying CS, NNR, and

IIA. It does not generalize further for two reasons. First, those strategy-proof voting procedures that are not regular do not have underlying social welfare functions. An example of such a case is the dictatorial voting procedure $v^{nm}(B) = f_S{}^i(B)$ where $f_S{}^i[B_i = (x \approx y \approx z)] = z$ and $f_S{}^i[B_i = (x \approx z\, y)] = x$. Inspection shows that no social welfare function u^{nm} satisfying CS, NNR, and IIA exists such that $\Psi_S[u^{nm}(B)] = f_T{}^i(B)$ for all $B \in \pi_m{}^n$. Second, if a social welfare function has a range that both strictly contains ρ_m and is contained in π_m, then, for some $B \in \pi_m{}^n$, $\Psi_S[u^{nm}(B)]$ will be a set with at least two elements. Therefore, because voting procedures have single element images, $\Psi_S[u^{nm}(B)]$ does not define a voting procedure.

Proof. Let $\mathscr{V}$ represent the collection of strategy-proof voting procedures v^{nm} and $\mathscr{U}$ represent the collection of social welfare functions u^{nm} that satisfy CS, NNR, and IIA. The subscript R indicates restriction of the collections $\mathscr{V}$ and $\mathscr{U}$ to regular voting procedures and regular social welfare functions, respectively. Similarly, a subscript S indicates restriction of the collections $\mathscr{V}$ and $\mathscr{U}$ to strict voting procedures and strict social welfare functions, respectively. By definition each $u^{nm} \in \mathscr{U}_R$ can be written as $\mu[\gamma(B)]$ where μ is a strict social welfare function and γ is a regular tie-breaking function. Clearly μ satisfies IIA, NNR, and CS, i.e., $\mu \in \mathscr{U}_S$.[10] Theorem 2 states that there exists a unique $\nu \in \mathscr{V}_S$ which is derived from μ. Thus, for all $C \in \rho_m{}^n$,

$$\Psi_S[\mu(C)] = \nu(C). \tag{43}$$

Define v^{nm} such that $v^{nm}(B) = \nu[\gamma(B)]$. Observe that v^{nm} is both regular and, by Lemma 9, strategy-proof, i.e., $v^{nm} \in \mathscr{V}_R$. Let $\gamma(B) = C$ for all $B \in \pi_m{}^n$. Substitution for C in (43) gives $\Psi_S\{\mu[\gamma(B)]\} = \nu[\gamma(B)]$ which simplifies to $\Psi_S[u^{nm}(B)] = v^{nm}(B)$. Therefore a $v^{nm} \in \mathscr{V}_R$ can be derived from every $u^{nm} \in \mathscr{U}_R$. Moreover, v^{nm} is uniquely derived from u^{nm} because $\Psi_S[u^{nm}(B)]$ is a single-valued function when $u^{nm} \in \mathscr{U}_R$.

By definition every $v^{nm} \in \mathscr{V}_R$ can be written $\nu[\gamma(B)]$ where ν is a strict voting procedure and γ is a regular tie-breaking function. Clearly, ν is strategy-proof, i.e., $\nu \in \mathscr{V}_S$. Theorem 2 guarantees that a unique $\mu \in \mathscr{U}_S$ exists such that (43) holds. Define u^{nm} such that $u^{nm}(B) = \mu[\gamma(B)]$. Lemma 10 implies that $u^{nm} \in \mathscr{U}_R$. Substitution into (43) gives

$$\Psi_S[u^{nm}(B)] = v^{nm}(B).$$

[10] Suppose $\mu \notin \mathscr{U}_S$. This implies that u^{nm} does not satisfy IIA, NNR, and CS over the domain $\rho_m{}^n$. Therefore, u^{nm} does not satisfy IIA and NNR over the domain $\pi_m{}^n$. Moreover, since $u^{nm}(B) = \mu[\gamma(B)]$ for all $B \in \pi_m{}^n$, if μ does not satisfy *CS* over $\rho_m{}^n$, then $u_m{}^n$ does not satisfy *CS* over the domain $\pi_m{}^n$. Therefore, if $u \in \mathscr{U}_R$, then $\mu \in \mathscr{U}_S$.

Therefore a $u^{nm} \in \mathscr{U}_R$ underlies every $v^{nm} \in \mathscr{V}_R$. Moreover, the $u^{nm} \in \mathscr{U}_R$ underlying each $v^{nm} \in \mathscr{V}_R$ is unique. This is shown by making minor changes in the uniqueness proof contained in Section 4.

Let u^* be any element $\mathscr{U}_R$ and let $v^* \in \mathscr{V}_R$ be the unique voting procedure derived from it. Since $\Psi_S[u^*(B)] = v^*(B)$, u^* is the unique element of $\mathscr{U}_R$ which underlies v^*. But every $v^{nm} \in \mathscr{V}_R$ has its unique $u^{nm} \in \mathscr{U}_R$ underlying it. Thus, the correspondence λ exists and is one-to-one. ∎

LEMMA 11. *Consider a committee $\langle I_n, S_m, u^{nm}\rangle$ where $n \geqslant 2$, $m \geqslant 3$, and u^{nm} is a social welfare function. Let γ be an arbitrary regular tie-breaking function and define the social welfare function μ^{nm} such that, for all $B \in \pi_m{}^n$,*

$$\mu^{nm}(B) = \gamma[u^{nm}(B)]. \tag{44}$$

If u^{nm} satisfies CS, NNR, *and* IIA, *then μ^{nm} has range contained in ρ_m and satisfies* CS, NNR, *and* IIA.

Proof. Assume that u^{nm} satisfies CS, NNR, and IIA. Equation (44) and the definition of γ directly imply that the range of μ^{nm} is contained in ρ_m. They also imply that μ^{nm} satisfies CS. Suppose μ^{nm} violates IIA: a $B \in \pi_m{}^n$, $C \in \pi_m{}^n$, and $U \subset S_m$ therefore exist such that $[\theta_U(B_1),..., \theta_U(B_n)] = [\theta_U(C_1),..., \theta_U(C_n)]$ and $\Psi_U[\mu^{nm}(B)] \neq \Psi_U[\mu^{nm}(C)]$. Nevertheless, $\Psi_U[u^{nm}(B)] = \Psi_U[u^{nm}(C)]$ because u^{nm} satisfies IIA. Moreover, $\Psi_U\{\gamma[u^{nm}(B)]\} = \Psi_U\{\gamma[u^{nm}(C)]\}$ because γ is regular. This contradicts the assumption that $\Psi_U[\mu^{nm}(B)] \neq \Psi_U[\mu^{nm}(C)]$.

Suppose μ^{nm} violates NNR: a $B = (B_1,..., B_i,..., B_n) \in \pi_m{}^n$, a $B_i' \in \pi_m$, $B_i \neq B_i'$, and a $x, y \in S_m$ exist such that yB_ix, $xB_i'y$ $x\bar{A}y$, and $y\bar{A}'x$ where $A = u^{nm}(B_1,..., B_i,..., B_n) = \mu^{nm}(B)$ and $A' = \mu^{nm}(B_1,..., B_i',..., B_n) = \mu^{nm}(B')$. In addition B_i and B_i' have the property that $\theta_U(B_i) = \theta_U(B_i')$ where $U = S_m - (x)$. Let $u^{nm}(B) = A^*$ and $u^{nm}(B') = A^{*\prime}$. Since $A = \gamma(A^*)$ and $A' = \gamma(A^{*\prime})$, consistency with the definition of γ implies that either $x\bar{A}^*y$ and $yA^{*\prime}x$ or xA^*y and $y\bar{A}^{*\prime}x$. Nevertheless, u^{nm} satisfies NNR. Application of NNR to $u^{nm}(B)$ and $u^{nm}(B')$ implies that if xA^*y, then $xA^{*\prime}y$ and if $x\bar{A}^*y$, then $x\bar{A}^{*\prime}y$. Thus, for μ^{nm} to violate NNR contradicts the assumption that u^{nm} satisfies NNR. ∎

THEOREM 3'. (Arrow). *Consider a committee $\langle I_n, S_m, u^{nm}\rangle$ where $n \geqslant 2$ and $m \geqslant 3$. The social welfare function u^{nm} satisfies* CS, NNR, *and* IIA *only if it is dictatorial.*

Proof. Assume that u^{nm} satisfies CS, NNR, and IIA. Lemmas 10 and 11 imply that an arbitrary regular tie-breaking function γ, a strict social welfare function μ satisfying CS, NNR, and IIA, and a tie-breaking function α exists such that

$$\gamma[u^{nm}(B)] = \mu[\alpha(B)] \tag{45}$$

for all $B \in \pi_m{}^n$. Pick $U = (x, y)$. Set $Q \in \rho_m$, the tie-breaking order for γ, such that $x\bar{Q}y$. Theorem 3 states that because μ satisfies CS, NNR, and IIA it is dictatorial. Assume that individual i is the dictator of μ.

Let γ' be a regular tie-breaking function with tie-breaking order $Q' \in \rho_m$ such that $y\bar{Q}'x$. As above, we can write

$$\gamma'[u^{nm}(B)] = \mu'[\alpha'(B)] \tag{46}$$

for all $B \in \pi_m{}^n$. Suppose $j \in I_n$ is the dictator for μ' where $j \neq i$. Consider a ballot set $C \in \pi_m{}^n$ such that $y\bar{C}_i x$ and $x\bar{C}_j y$. The assumed dictators for μ and μ' imply that $y\bar{A}_\gamma x$ and $x\bar{A}_\gamma' y$ where $A_\gamma = \gamma(A) = \gamma[u^{nm}(C)]$ and $A_\gamma' = \gamma'(A) = \gamma'[u^{nm}(C)]$. This, however, is a contradiction. Recall that γ and γ', respectively, have tie-breaking orders Q and Q' such that $x\bar{Q}y$ and $y\bar{Q}'x$. Therefore, $y\bar{A}_\gamma x$ implies $y\bar{A}x$ while $x\bar{A}'y$ implies $x\bar{A}y$. Thus, $i = j$, i.e., μ and μ' have the same dictator.

Suppose in Eq. (45) individual i is the dictator for μ, but not for u^{nm}. Therefore, a $C \in \pi_m{}^n$ exists such that for some $x, y \in S_m$, $y\bar{C}_i x$ and $\sim y\bar{A}x$ where $A = u^{nm}(C)$. Since μ is dictatorial, $y\bar{A}^*x$ where $A^* = \mu[\alpha(C)]$. Without loss of generality assume that the x and y of this paragraph are identical to the x and y of the preceding two paragraphs.[11] Recall that Q, the tie-breaking order for γ, has the property that $x\bar{Q}y$. Let $A' = \gamma(A) = \gamma[u^{nm}(C)]$. Therefore, $\sim y\bar{A}x$ implies that $x\bar{A}'y$. But $A' = \gamma[u^{nm}(C)] = \mu[\alpha(C)] = A^*$ and, from above, $y\bar{A}^*x$. This contradicts the result that $x\bar{A}'y$. Therefore individual i must be the dictator of u^{nm}. ▌

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[11] There would be a loss of generality here if I had not shown above that, given γ and γ', both μ and μ' have the same dictator.

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